

QUANTUM CURVE IN q -OSCILLATOR MODEL

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ABSTRACT. A lattice model of interacting q -oscillators, proposed in [V. Bazhanov, S. Sergeev, arXiv:hep-th/0509181], is the quantum mechanical integrable model in 2+1 dimensional space-time. Its layer-to-layer transfer-matrix is a polynomial of two spectral parameters, it may be regarded in the terms of quantum groups both as a sum of $sl(N)$ transfer matrices of a chain of length M and as a sum of $sl(M)$ transfer matrices of a chain of length N for reducible representations. The aim of this paper is to derive the Bethe Ansatz equations for the q -oscillator model entirely in the framework of 2+1 integrability providing the evident rank-size duality.

INTRODUCTION

The q -oscillator lattice model was formulated recently in [1,2]. It describes a system of interacting q -oscillators situated in the vertices of two dimensional lattice, and therefore it is the quantum mechanical system in $2 + 1$ dimensional space-time in the same way, as a chain of interacting particles (or spins) is regarded as a model in $1 + 1$ dimensional space-time. Formulation of the q -oscillator model provides a definition of a layer-to-layer transfer matrix as a polynomial of two spectral parameters. This transfer matrix may be interpreted in the terms of quantum inverse scattering method and quantum groups, so that both sizes of the two dimensional lattice may be interpreted as either a length of an effective chain or as symmetry group's rank. This was called in [1] the “rank-size” duality. The appearance of a complete set of fundamental transfer matrices for $\mathcal{U}_q(\widehat{sl})$ series is a signal that the layer-to-layer transfer matrix of q -oscillator model is closely related to Bethe Ansatz in the form of generalized Baxter's “T-Q” equations. The subject of this paper is the derivation of such equations in the framework of 2+1 dimensional integrability.

Below in this Introduction we formulate the answer, i.e. we give an explicit form of “T-Q” equations in the terms of a given layer-to-layer transfer matrix. To do this, we

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need to repeat the structure of the layer-to-layer transfer matrix for q -oscillator model in more details.

The q -oscillator model describes a system of interacting q -oscillators \mathcal{H}_v ,

$$(1) \quad \mathcal{H}_v : \mathbf{x}_v \mathbf{y}_v = 1 - q^{2+2\mathbf{h}_v}, \quad \mathbf{y}_v \mathbf{x}_v = 1 - q^{2\mathbf{h}_v}, \quad \mathbf{x}_v q^{\mathbf{h}_v} = q^{\mathbf{h}_v+1} \mathbf{x}_v, \quad \mathbf{y}_v q^{\mathbf{h}_v} = q^{\mathbf{h}_v-1} \mathbf{y}_v,$$

situated in the vertices v of two dimensional square lattice of sizes $N \times M$. Index v stands for a coordinate of a vertex. Oscillators from different vertices commute (what is called “locality”), the whole *algebra of observables* is thus $\mathcal{H}^{\otimes NM}$, and the vertex index v corresponds to the number of component of the tensor power. In this paper we imply mostly the Fock space representation \mathcal{F} of q -oscillators.

The two dimensional lattice may be identified with a layer (or a section) of three dimensional cubic lattice, further we call it either the layer or the auxiliary lattice.

Auxiliary matrices $L_{\alpha,\beta}[\mathcal{H}_v]$, acting in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{F}_v$, were introduced in [1]. The *layer* transfer matrix $\mathbf{T}(u, v)$ may be constructed as a trace of an $2d$ ordered product of auxiliary matrices $L[\mathcal{H}_v]$. The transfer matrix is a polynomial of two spectral parameters,

$$(2) \quad \mathbf{T}(u, v) = \sum_{n=0}^N \sum_{m=0}^M u^n v^m \mathbf{t}_{n,m},$$

its coefficients $\mathbf{t}_{n,m} \in \mathcal{H}^{\otimes NM}$ form a complete commutative set. Matrices $L[\mathcal{H}_v]$ depend on some extra \mathbb{C} -valued free parameters, for their generic values the model is inhomogeneous. The layer transfer matrix (2) may be identically rewritten in two ways,

$$(3) \quad \mathbf{T}(u, v) \equiv \sum_{n=0}^N u^n T_{\omega_n}^{(sl_N)}(v) \equiv \sum_{m=0}^M v^m T_{\omega_m}^{(sl_M)}(u),$$

where

$$(4) \quad T_{\omega_n}^{(sl_N)}(v) = \sum_{m=0}^M v^m \mathbf{t}_{n,m}$$

is the $2d$ transfer matrix for $\mathcal{U}_q(\widehat{sl_N})$ *chain* of the length M , corresponding to the fundamental representation π_{ω_n} in the auxiliary space (here ω_n stand for the fundamental weights of A_{N-1} , π_{ω_0} and π_{ω_N} are two scalar representations, $T_0^{(sl_N)}$ and $T_N^{(sl_N)}$ may be written explicitly). The same layer transfer matrix $\mathbf{T}(u, v)$ may be rewritten as the sum of $\mathcal{U}_q(\widehat{sl_M})$ transfer matrices

$$(5) \quad T_{\omega_m}^{(sl_M)}(u) = \sum_{n=0}^N u^n \mathbf{t}_{n,m}$$

for the length N chain (the last part of (3)).

The result of this paper is the derivation of the dual Bethe Ansatz equations for the q -oscillator model. They may be formulated as follows. Let normalized transfer matrices be

$$(6) \quad \tau_m^{(sl_M)}(u) = T_{\omega_m}^{(sl_M)}(-(-q)^m u), \quad \tau_n^{(sl_N)}(v) = T_{\omega_n}^{(sl_N)}(-(-q)^n v).$$

And let \mathbb{C} -numerical parameters of q -oscillator lattice are inhomogeneous enough.

Then “T-Q” equation for sl_M is

$$(7) \quad \sum_{m=0}^M (-v)^m \tau_m^{(sl_M)}(u) Q(q^{2m} u) = 0.$$

The statement is that if $\mathbf{t}_{n,m}$ take their eigenvalues, then there exist M special values v_1, \dots, v_M of v , such that corresponding $Q_1(u), \dots, Q_M(u)$ in (7) are polynomials¹. Degrees of the polynomials are uniquely defined by certain occupation numbers of oscillators.

In its turn, equivalent “T-Q” equation for sl_N is

$$(8) \quad \sum_{n=0}^N (-u)^n \tau_n^{(sl_N)}(v) \overline{Q}(q^{2n} v) = 0.$$

If $\mathbf{t}_{n,m}$ take their eigenvalues, then there exist N special values u_1, \dots, u_N of u , such that corresponding $\overline{Q}_1(v), \dots, \overline{Q}_N(v)$ in (8) are polynomials. All the other forms of nested Bethe Ansatz equations follow from (7) or (8).

Polynomials $Q(u)$ and $\overline{Q}(v)$ may be denoted in the quantum mechanical way as “wave functions” of states $\langle Q|$ and $|\overline{Q}\rangle$:

$$(9) \quad Q(u) = \langle Q|u\rangle \quad \text{and} \quad \overline{Q}(v) = \langle v|\overline{Q}\rangle,$$

where $|u\rangle$ and $\langle v|$ serve the simple Weyl algebra \mathcal{W} : $\mathbf{u} \mathbf{v} = q^2 \mathbf{v} \mathbf{u}$,

$$(10) \quad \begin{aligned} \mathbf{u}|u\rangle &= |u\rangle u, \quad \mathbf{v}|u\rangle = |q^2 u\rangle v, \\ \text{and} \\ \langle v|\mathbf{u} &= u\langle q^2 v|, \quad \langle v|\mathbf{v} = v\langle v|. \end{aligned}$$

Let

$$(11) \quad J(\mathbf{u}, \mathbf{v}) = \sum_{n=0}^N \sum_{m=0}^M (-q)^{-nm} (-\mathbf{u})^n (-\mathbf{v})^m t_{n,m}.$$

Then (7) and (8) are correspondingly

$$(12) \quad \langle Q| J(\mathbf{u}, \mathbf{v}) |u\rangle = \langle v| J(\mathbf{u}, \mathbf{v}) |\overline{Q}\rangle = 0.$$

¹parameter v in the “ u -shift” equation (7) is irrelevant since a re-scaling $Q(u) \rightarrow u^\nu Q(u)$ changes it.

The formulation of the q -oscillator model and definition of $\mathbf{T}(u, v)$ are locally $3d$ invariant, the quantum group interpretation (3) is the secondary one. In this paper we will derive (11) without any quantum group technique.

To explain our method, we need to comment a little the classical limit.

In the classical limit $q \rightarrow 1$, the local q -oscillator generators become the classical dynamical variables, the q -oscillator model becomes a model of classical mechanics, quantum evolution operators become Baecklund transformations for the dynamical variables. In particular, $T(u, v)$ may be understood as a partition function of a completely inhomogeneous free fermion six-vertex model on the square lattice (but it should not be regarded as a model of statistical mechanics). In its turn, $J(u, v)$ becomes a free fermion determinant (the sign $(-)^{nm+n+m}$ counts the number of fermionic loops). There exists a well known formula in the theory of two dimensional free fermion models, relating T and J :

$$(13) \quad T(u, v) = \frac{1}{2} (J(-u, v) + J(u, -v) + J(-u, -v) - J(u, v)) .$$

In the classical limit, equation $J(u, v) = 0$ defines the spectral curve. Dynamical variables may be expressed in the terms of θ -functions on the Jacobian of the spectral curve. The sequence of Baecklund transformations, which is the “discrete time” in the classical model, is a sequence of linear shifts of a point on the Jacobian. The classical model was formulated and solved by I. G. Korepanov [3].

Classical integrability is based on an auxiliary linear problem. Equation $J(u, v) = 0$ is the condition of the existence of a solution of the linear problem. Our point is that in quantum $q \neq 1$ case, the linear problem is still the basic concept of the solvability. Quantum $\mathbf{J}(\mathbf{u}, \mathbf{v})$ is a well defined determinant of an operator-valued matrix, and $\mathbf{J}(\mathbf{u}, \mathbf{v})|\Psi\rangle = 0$ is again the condition of the existence of a solution of a quantum linear problem. The polynomial structure of e.g. $\langle v|\Psi\rangle$ follows from a more detailed consideration of the quantum linear problem in a special basis of diagonal “quantum Baker-Akhiezer function” (related to a quantum separation of variables).

The structure of the paper is the following. In the first section we recall briefly some basic notions of the classical model [3]: the linear problem, spectral curve and details of the combinatorial representation of the spectral curve. In the second section we repeat the definition of the quantum model and its integrability [1, 2]. In particular, our definition of the spectral parameters differs from that of [1]. Quantum linear problem, derivation of (11) and properties of various forms of (12) are given in the third section. The fourth section includes an example.

form, the right hand side of Fig. 1 in matrix notations, is

$$(15) \quad \begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \end{pmatrix} = X[A_v] \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix}, \quad \text{where} \quad X[A_v] \stackrel{\text{def}}{=} \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}.$$

1.2. Korepanov's equation. Equations of motion in an integrable model arise as an associativity condition of its linear problem. To derive their local form, consider a vertex of three dimensional lattice (not necessarily the cubic one), and sections of it by two auxiliary planes, as it is shown in Fig. 2. From three dimensional point of view, the auxiliary linear variables belong to the faces of 3d lattice, while the dynamical variables A_v belong to the edges of the 3d lattice. Therefore A_v are distinguished from A'_v , but linear variables on outer edges $\psi_\alpha, \dots, \psi'_\gamma$, in both top and bottom auxiliary planes are identified. Consider the bottom plane first. The linear problem rule (15) may be applied

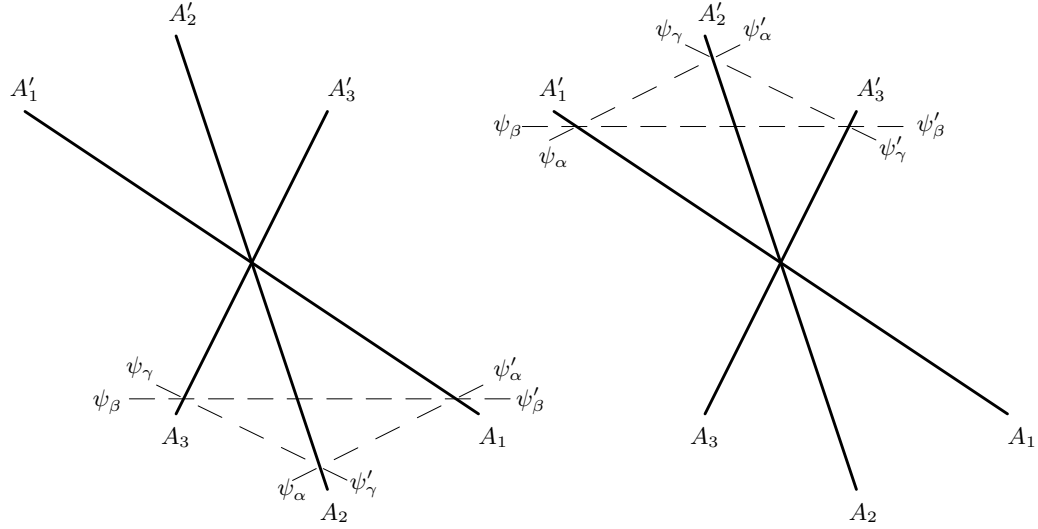


FIGURE 2. Left and right hand sides of Korepanov equation.

three times for excluding internal edges, as the result one obtains an expression of the “primed” linear variables in the terms of “unprimed”:

$$(16) \quad \begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \\ \psi'_\gamma \end{pmatrix} = X_{\alpha,\beta}[A_1] \cdot X_{\alpha,\gamma}[A_2] \cdot X_{\beta,\gamma}[A_3] \cdot \begin{pmatrix} \psi_\alpha \\ \psi_\beta \\ \psi_\gamma \end{pmatrix},$$

where (cf. the ordering of α, β, γ in column vectors)

$$(17) \quad X_{\alpha, \beta}[A_1] = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{\alpha, \gamma}[A_2] = \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}, \quad X_{\beta, \gamma}[A_3] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}$$

The top plane of Fig. 2 may be considered in the same way,

$$(18) \quad \begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \\ \psi'_\gamma \end{pmatrix} = X_{\beta, \gamma}[A'_3] \cdot X_{\alpha, \gamma}[A'_2] \cdot X_{\alpha, \beta}[A'_1] \cdot \begin{pmatrix} \psi_\alpha \\ \psi_\beta \\ \psi_\gamma \end{pmatrix},$$

where the matrices $X_{\#, \#}$ are given by (17) with $A'_\mathbf{v} = (a'_\mathbf{v}, b'_\mathbf{v}, c'_\mathbf{v}, d'_\mathbf{v})$.

The associativity condition of linear problems (16) and (18) is the Korepanov equation

$$(19) \quad X_{\alpha, \beta}[A_1] \cdot X_{\alpha, \gamma}[A_2] \cdot X_{\beta, \gamma}[A_3] = X_{\beta, \gamma}[A'_3] \cdot X_{\alpha, \gamma}[A'_2] \cdot X_{\alpha, \beta}[A'_1],$$

relating the set of 12 variables $A_\mathbf{v}$ with the set of 12 variables $A'_\mathbf{v}$, $\mathbf{v} = 1, 2, 3$. Equation (19) describes a single 3d vertex. Equations of motion for three dimensional integrable model is the collection of equations (19) for all vertices of the 3d lattice.

Korepanov equation needs a very important comment. Matrices $X_{\#, \#}[A_\mathbf{v}]$ by definition (17) act in the direct sum of one-dimensional vector spaces labelled by the indices α, β, γ , etc. The matrix $X_{\alpha, \beta}[A_1]$ in the block (α, β) coincides with $X[A_1]$ (15), and in the block (γ, \dots) it is the unity matrix. In what follows, such “direct sum” imbedding of 2×2 matrices X into higher dimensional unity matrices will always be implied.

1.3. Linear problem with periodical boundary conditions. Korepanov’s solution of the equations of motion is based on the solution of the linear problem for the whole auxiliary lattice. Consider the square lattice with the sizes $N \times M$. Let the lines of the lattice are enumerated by

$$(20) \quad \alpha_n \quad \text{and} \quad \beta_m, \quad n = 1, 2, \dots, N, \quad m = 1, 2, \dots, M.$$

A fragment of the auxiliary lattice is shown in Fig. 3. Notations for vertex and auxiliary variables for $(n, m)^{\text{th}}$ vertex of the plane are shown in Fig. 4. The local auxiliary linear problem (15) for $(n, m)^{\text{th}}$ vertex takes the form

$$(21) \quad \begin{pmatrix} \psi_{\alpha_n}^{(m-1)} \\ \psi_{\beta_m}^{(n-1)} \end{pmatrix} = X[A_{n, m}] \cdot \begin{pmatrix} \psi_{\alpha_n}^{(m)} \\ \psi_{\beta_m}^{(n)} \end{pmatrix}, \quad n = 1, \dots, N, \quad m = 1, \dots, M.$$

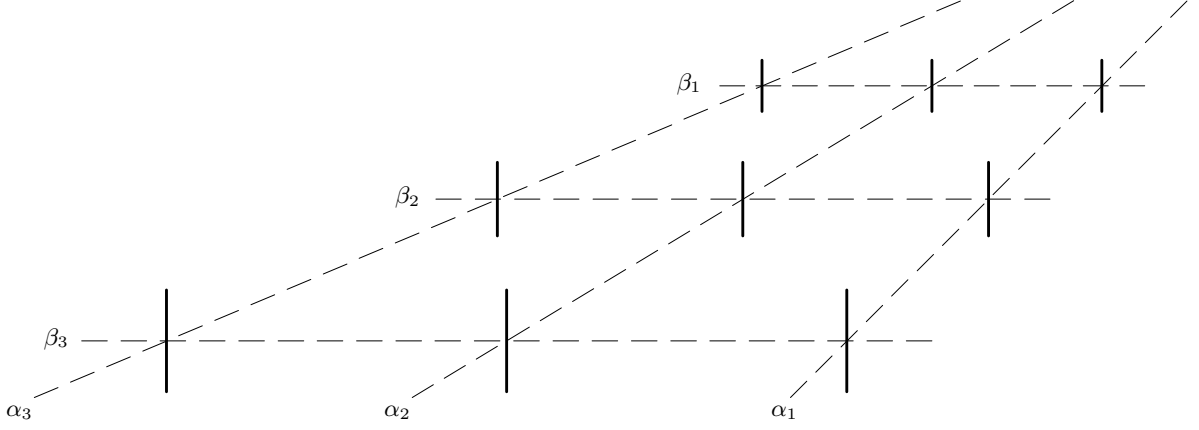
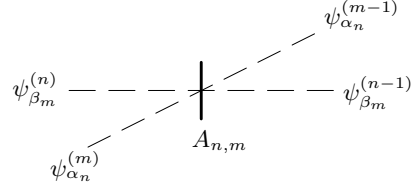


FIGURE 3. A fragment of the auxiliary square lattice.

The linearity of the whole set of (21) with respect to ψ -s makes it possible to define the

FIGURE 4. Notations for $(n, m)^{\text{th}}$ vertex of the auxiliary lattice.

quasi-periodical boundary conditions for them:

$$(22) \quad \psi_{\alpha_n}^{(m+M)} = u \psi_{\alpha_n}^{(m)} \quad \text{and} \quad \psi_{\beta_m}^{(n+N)} = v \psi_{\beta_m}^{(n)},$$

where u and v are \mathbb{C} -valued spectral parameters.

Linear equations (21) may be iterated for the whole lattice as follows. Let

$$(23) \quad \psi_{\alpha}^{(m)} = \begin{pmatrix} \psi_{\alpha_1}^{(m)} \\ \psi_{\alpha_2}^{(m)} \\ \vdots \\ \psi_{\alpha_N}^{(m)} \end{pmatrix}, \quad \psi_{\beta}^{(n)} = \begin{pmatrix} \psi_{\beta_1}^{(n)} \\ \psi_{\beta_2}^{(n)} \\ \vdots \\ \psi_{\beta_M}^{(n)} \end{pmatrix}.$$

Then the repeated use of (21) gives

$$(24) \quad \begin{pmatrix} \psi_{\alpha}^{(0)} \\ \psi_{\beta}^{(0)} \end{pmatrix} = \mathbb{X}_{\alpha,\beta} \begin{pmatrix} \psi_{\alpha}^{(M)} \\ \psi_{\beta}^{(N)} \end{pmatrix},$$

where, in the terms of matrix imbedding discussed right after (19), the $(N+M) \times (N+M)$ monodromy matrix $\mathbb{X}_{\alpha,\beta}$ may be written as

$$(25) \quad \mathbb{X}_{\alpha,\beta} = \prod_n^{\curvearrowright} \prod_m^{\curvearrowright} X_{\alpha_n,\beta_m}[A_{n,m}],$$

where

$$(26) \quad \prod_n^{\curvearrowright} f_n \stackrel{\text{def}}{=} f_1 f_2 \dots f_{N-1} f_N, \quad \prod_m^{\curvearrowright} f_m \stackrel{\text{def}}{=} f_1 f_2 \dots f_{M-1} f_M.$$

The boundary conditions (22) give $\psi_{\alpha}^{(M)} = u\psi_{\alpha}^{(0)}$, $\psi_{\beta}^{(N)} = v\psi_{\beta}^{(0)}$, so that (24) becomes

$$(27) \quad \left(1 - \mathbb{X}_{\alpha,\beta} \cdot \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right) \begin{pmatrix} \psi_{\alpha}^{(0)} \\ \psi_{\beta}^{(0)} \end{pmatrix} = 0.$$

The whole linear problem has a solution if and only if

$$(28) \quad J(u, v) \stackrel{\text{def}}{=} \det \left(1 - \mathbb{X}_{\alpha,\beta} \cdot \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right)$$

is zero. Equation $J(u, v) = 0$ defines the spectral curve for the model, equations of motion (19) for the whole three dimensional lattice have an exact solution in the terms of θ -functions on the Jacobian of the spectral curve [3].

1.4. Free fermion model. The determinant (28) has the very well known combinatorial representation. Usual way to derive it is to define the determinant in the terms of the Grassmanian integration and then to turn from normal symbols to matrix elements.

Let in this subsection ψ and $\bar{\psi}$ be the Grassmanian variables with the integration rules $\int d\psi = \int d\bar{\psi} = 0$ and $\int \psi d\psi = \int \bar{\psi} d\bar{\psi} = 1$. Then the determinant (28) may be written as

$$(29) \quad J(u, v) = \int e^{\mathcal{A}[\bar{\psi}, \psi]} \mathcal{D}\bar{\psi} \mathcal{D}\psi,$$

where the “action” is

$$(30) \quad \mathcal{A} = \sum_{n,m=1}^{N,M} \left\{ \left(\bar{\psi}_{\alpha_n}^{(m-1)}, \bar{\psi}_{\beta_m}^{(n-1)} \right) \cdot X[A_{n,m}] \cdot \begin{pmatrix} \psi_{\alpha_n}^{(m)} \\ \psi_{\beta_m}^{(n)} \end{pmatrix} + \psi_{\alpha_n}^{(m)} \bar{\psi}_{\alpha_n}^{(m)} + \psi_{\beta_m}^{(n)} \bar{\psi}_{\beta_m}^{(n)} \right\},$$

and the measure is

$$(31) \quad \mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_{n,m=1}^{N,M} d\bar{\psi}_{\alpha_n}^{(m)} d\psi_{\alpha_n}^{(m)} d\bar{\psi}_{\beta_m}^{(n)} d\psi_{\beta_m}^{(n)} .$$

Spectral parameters appear in (29) via

$$(32) \quad \bar{\psi}_{\alpha_n}^{(0)} = u\bar{\psi}_{\alpha_n}^{(M)} \quad \text{and} \quad \bar{\psi}_{\beta_m}^{(0)} = v\bar{\psi}_{\beta_m}^{(N)} .$$

In the terms of Grassmanian variables, the exponent of a quadratic form is a normal symbol of some operator L ,

$$(33) \quad \exp \left\{ (\bar{\psi}_\alpha, \bar{\psi}_\beta) \cdot X[A_v] \cdot \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix} \right\} \stackrel{\text{def}}{=} \langle \bar{\psi}_\alpha, \bar{\psi}_\beta | L_{\alpha,\beta}[A_v] | \psi_\alpha, \psi_\beta \rangle .$$

The fermionic coherent states are defined by

$$(34) \quad |\psi\rangle = |0\rangle + |1\rangle\psi, \quad \langle\bar{\psi}| = \langle 0| + \bar{\psi}\langle 1| ,$$

and the extra summands in (30) correspond to the unity operators

$$(35) \quad 1 = \int |\psi\rangle e^{\psi\bar{\psi}} d\bar{\psi} d\psi \langle\bar{\psi}| .$$

It is important to note that the indices of the operator $L_{\alpha,\beta}[A_v]$ (33) label copies of two-dimensional vector spaces (34) $\mathbb{C}^2 \ni x|0\rangle + y|1\rangle$. Thus, $L_{\alpha,\beta}$ acts in the tensor *product* of two-dimensional vector spaces, while $X_{\alpha,\beta}$ acts in the tensor *sum* of one-dimensional vector spaces. In the basis of the fermionic states

$$(36) \quad |n_\alpha, n_\beta\rangle = \left(|0, 0\rangle, |1, 0\rangle, |0, 1\rangle, |1, 1\rangle \right) ,$$

operator $L_{\alpha,\beta}$ (33) is 4×4 matrix

$$(37) \quad L_{\alpha,\beta}[A_v] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_v & b_v & 0 \\ 0 & c_v & d_v & 0 \\ 0 & 0 & 0 & z_v \end{pmatrix} , \quad \text{where} \quad z_v \stackrel{\text{def}}{=} b_v c_v - a_v d_v .$$

Besides, the Korepanov equation (19) is the equality of the exponents of the normal symbol form of the local Yang-Baxter equation

$$(38) \quad L_{\alpha,\beta}[A_1]L_{\alpha,\gamma}[A_2]L_{\beta,\gamma}[A_3] = L_{\beta,\gamma}[A'_3]L_{\alpha,\gamma}[A'_2]L_{\alpha,\beta}[A'_1] ,$$

since

$$(39) \quad \langle\bar{\psi}|L_{\alpha,\beta}L_{\alpha,\gamma}L_{\beta,\gamma}|\psi\rangle = \exp \{ \bar{\psi} \cdot X_{\alpha,\beta}X_{\alpha,\gamma}X_{\beta,\gamma} \cdot \psi \} \quad \text{etc.}$$

Turn now to the expression of the determinant (28) in the terms of operators L . Let $2^{N+M} \times 2^{N+M}$ matrix $\mathbb{L}_{\alpha,\beta}$ be the ordered product of local L -s:

$$(40) \quad \mathbb{L}_{\alpha,\beta} = \prod_n^{\sim} \prod_m^{\sim} L_{\alpha_n,\beta_m}[A_{n,m}] .$$

This is related to the monodromy matrix (25) by means of (cf. (33))

$$(41) \quad \exp \left\{ (\bar{\psi}_{\alpha}, \bar{\psi}_{\beta}) \cdot \mathbb{X}_{\alpha,\beta} \cdot \begin{pmatrix} \psi_{\alpha} \\ \psi_{\beta} \end{pmatrix} \right\} = \langle \bar{\psi}_{\alpha}, \bar{\psi}_{\beta} | \mathbb{L}_{\alpha,\beta} | \psi_{\alpha}, \psi_{\beta} \rangle .$$

Define now the boundary matrices for $\mathbb{L}_{\alpha,\beta}$,

$$(42) \quad D(u) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} , \quad D_{\alpha}(u) = \prod_n D_{\alpha_n}(u) , \quad D_{\beta}(v) = \prod_m D_{\beta_m}(v) ,$$

and let

$$(43) \quad T(u, v) = \text{Trace}_{\alpha,\beta} \left(D_{\alpha}(u) D_{\beta}(v) \mathbb{L}_{\alpha,\beta} \right) .$$

By the construction, $T(u, v)$ is the partition function for a free-fermion lattice model with the inhomogeneous Boltzmann weights – matrix elements of $L_{\alpha_n,\beta_m}[A_{n,m}]$ – and u, v -boundary conditions. It is the polynomial of u and v :

$$(44) \quad T(u, v) = \sum_{n=0}^N \sum_{m=0}^M u^n v^m t_{n,m} .$$

Sometimes a pure combinatorial representation of the partition function is very useful. Any monomial in $T(u, v)$ (43) corresponds to a non-self-intersecting path on the toroidal lattice. A path may go through a vertex in one of five different ways as it is shown in Fig. 5 (or do not go through at all). A factor f_v is associated with each variant, these factors are the matrix elements of $L_{\alpha,\beta}[A_v]$ (37). A monomial in $T(u, v)$, corresponding to path C , is

$$(45) \quad t_C = \prod_{\text{along path } C} f_v .$$

Any non-self-intersecting path on the toroidal lattice has a homotopy class

$$(46) \quad w(C) = nA + mB ,$$

where A is the toroidal cycle along the α -lines, and B is the toroidal cycle along the β -lines of Fig. 3. Then the element $t_{n,m}$ of (44) is

$$(47) \quad t_{n,m} = \sum_{C : w(C)=nA+mB} t_C$$

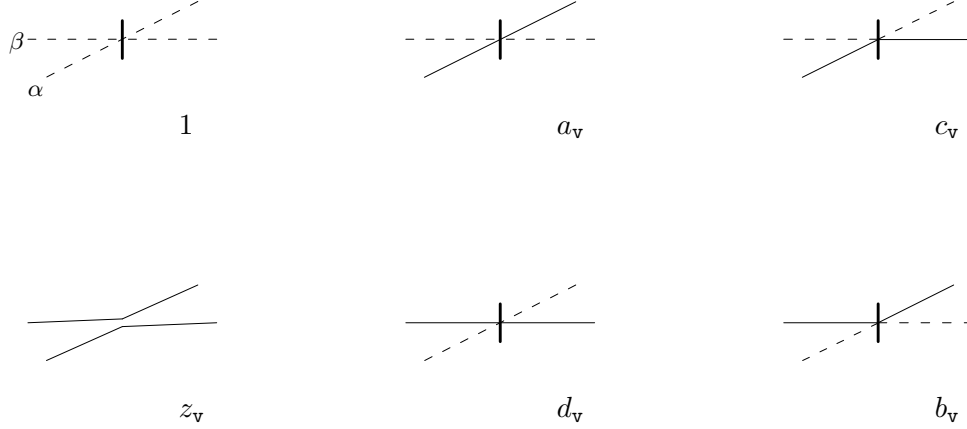


FIGURE 5. Six variants of bypassing the vertex. Vertex factors f_v are matrix elements of L . Note, in the variant z_v the path is not self-intersecting.

The determinant $J(u, v)$ (28) is related to $t_{n,m}$ via

$$(48) \quad J(u, v) = \sum_{n=0}^N \sum_{m=0}^M (-)^{nm+n+m} u^n v^m t_{n,m} ,$$

where the sign $(-)^{nm+n+m}$ counts the number of fermionic loops on the toroidal square lattice. The determinant J may be expressed in the terms of T and vice versa:

$$(49) \quad \begin{aligned} J(u, v) &= \frac{1}{2} (T(-u, -v) + T(-u, v) + T(u, -v) - T(u, v)) , \\ T(u, v) &= \frac{1}{2} (J(-u, -v) + J(-u, v) + J(u, -v) - J(u, v)) . \end{aligned}$$

The last equality is very well known in the two dimensional free-fermion model as the formula relating the lattice partition function and fermionic determinant.

Now we may finish the collection of notions and definitions of the Korepanov model of classical integrable dynamics on three dimensional lattice and proceed to the description of their quantum analogues.

2. QUANTUM MODEL

In the previous section, we did not pay any attention to the structure of vertex variables A_v (14), they were defined simply as the list of elements of X (15). The key point for the quantization of the model is that it is possible to define a local Poisson structure on

A_v [1] such that the transformation

$$(50) \quad A_1 \otimes A_2 \otimes A_3 \rightarrow A'_1 \otimes A'_2 \otimes A'_3 ,$$

defined by Korepanov equation (19), is a symplectic map. Symplectic structure admits an immediate quantization. We will skip here all the details and proceed directly to the Ansatz for quantized A_v , $X[A_v]$ and $L[A_v]$. The aim of this section is just to give precise definition of $\mathbf{T}(u, v)$ (2).

2.1. Quantum Korepanov and Tetrahedron equations. The local q -oscillator algebra \mathcal{H} is defined by (1). The Fock space \mathcal{F} representation for q -oscillators corresponds to

$$(51) \quad \text{Spec}(\mathbf{h}_v) = 0, 1, 2, \dots \quad \forall v .$$

Quantized dynamical variables (14) are the q -oscillator generators and a pair of \mathbb{C} -valued parameters, $\mathcal{A}_v \sim (\mathcal{H}_v; \lambda_v, \mu_v)$:

$$(52) \quad \mathcal{A}_v = (a_v = \lambda_v q^{\mathbf{h}_v}, b_v = \mathbf{y}_v, c_v = -q^{-1} \lambda_v \mu_v \mathbf{x}_v, d_v = \mu_v q^{\mathbf{h}_v}) .$$

Quantized X (15) and L (37) are given by

$$(53) \quad X[\mathcal{A}_v] = \begin{pmatrix} \lambda_v q^{\mathbf{h}_v} & \mathbf{y}_v \\ -q^{-1} \lambda_v \mu_v \mathbf{x}_v & \mu_v q^{\mathbf{h}_v} \end{pmatrix}$$

and

$$(54) \quad L_{\alpha, \beta}[\mathcal{A}_v] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_v q^{\mathbf{h}_v} & \mathbf{y}_v & 0 \\ 0 & -q^{-1} \lambda_v \mu_v \mathbf{x}_v & \mu_v q^{\mathbf{h}_v} & 0 \\ 0 & 0 & 0 & -q^{-1} \lambda_v \mu_v \end{pmatrix} .$$

One may verify directly, the quantum Korepanov equation

$$(55) \quad X_{\alpha, \beta}[\mathcal{A}_1] X_{\alpha, \gamma}[\mathcal{A}_2] X_{\beta, \gamma}[\mathcal{A}_3] \mathbf{R} = \mathbf{R} X_{\beta, \gamma}[\mathcal{A}_3] X_{\alpha, \gamma}[\mathcal{A}_2] X_{\alpha, \beta}[\mathcal{A}_1]$$

is equivalent to the auxiliary tetrahedron equation (the quantum local Yang-Baxter equation)

$$(56) \quad L_{\alpha, \beta}[\mathcal{A}_1] L_{\alpha, \gamma}[\mathcal{A}_2] L_{\beta, \gamma}[\mathcal{A}_3] \mathbf{R} = \mathbf{R} L_{\beta, \gamma}[\mathcal{A}_3] L_{\alpha, \gamma}[\mathcal{A}_2] L_{\alpha, \beta}[\mathcal{A}_1] .$$

Here the intertwining operator $\mathbf{R} = \mathbf{R}_{123}$ acts in the tensor product of representation spaces $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3$ of three q -oscillators $\mathcal{H}_{1,2,3}$. Parameters λ_v, μ_v of \mathcal{A}_v are the parameters of \mathbf{R} . Both equations (55) and (56) are equivalent to the following set of six equations:

$$(57) \quad \begin{aligned} \mathbf{R} q^{\mathbf{h}_2} \mathbf{x}_1 &= \frac{\lambda_2}{\lambda_3} \left(q^{\mathbf{h}_3} \mathbf{x}_1 - \frac{q}{\lambda_1 \mu_3} q^{\mathbf{h}_1} \mathbf{x}_2 \mathbf{y}_3 \right) \mathbf{R}, & \mathbf{R} \mathbf{x}_2 &= \left(\mathbf{x}_1 \mathbf{x}_3 + \frac{q^2}{\lambda_1 \mu_3} q^{\mathbf{h}_1 + \mathbf{h}_3} \mathbf{x}_2 \right) \mathbf{R}, \\ \mathbf{R} q^{\mathbf{h}_2} \mathbf{x}_3 &= \frac{\mu_2}{\mu_1} \left(q^{\mathbf{h}_1} \mathbf{x}_3 - \frac{q}{\lambda_1 \mu_3} q^{\mathbf{h}_3} \mathbf{y}_1 \mathbf{x}_2 \right) \mathbf{R}, & \mathbf{R} \mathbf{y}_2 &= \left(\mathbf{y}_1 \mathbf{y}_3 + \lambda_1 \mu_3 q^{\mathbf{h}_1 + \mathbf{h}_3} \mathbf{y}_2 \right) \mathbf{R}, \\ \mathbf{R} q^{\mathbf{h}_1 + \mathbf{h}_2} &= q^{\mathbf{h}_1 + \mathbf{h}_2} \mathbf{R}, & \mathbf{R} q^{\mathbf{h}_2 + \mathbf{h}_3} &= q^{\mathbf{h}_2 + \mathbf{h}_3} \mathbf{R}. \end{aligned}$$

Classical equations (19) and (38) follow from (55) and (56) in the $q \rightarrow 1$ limit of the well defined automorphism $\mathcal{A}'_v = \mathbf{R} \mathcal{A}_v \mathbf{R}^{-1}$. For irreducible representations of \mathcal{H}_v , \mathbf{R} is defined uniquely, its matrix elements for the Fock space representation are given in [1]. Remarkably, Fig. 2 may be used for the graphical representation both of classical and quantum equations, in the quantum case the solid $3d$ cross in Fig. 2 stands for \mathbf{R} .

Integrable model of quantum mechanics may be formulated purely in the terms of matrices L (54). The intertwiner \mathbf{R} is related to evolution operators, it is another subject and we will not consider it here. Below we recall the definition of integrable model of quantum mechanics from [1, 2].

2.2. Transfer matrix \mathbf{T} . For the square lattice $N \times M$ of the previous section, define the “monodromy” of quantized L -s literally by (40):

$$(58) \quad \mathbb{L}_{\alpha, \beta} = \prod_n \prod_m L_{\alpha_n, \beta_m} [\mathcal{A}_{n,m}],$$

and its trace (cf. (43))

$$(59) \quad \mathbf{T}(u, v) = \text{Trace}_{\alpha, \beta} \left(D_{\alpha}(u) D_{\beta}(v) \mathbb{L}_{\alpha, \beta} \right),$$

where boundary matrices D are defined by (42). To distinguish the classical and quantum cases, we use the boldface letters for the quantized \mathbf{T} and its decomposition (44):

$$(60) \quad \mathbf{T}(u, v) = \sum_{n=0}^N \sum_{m=0}^M u^n v^m \mathbf{t}_{n,m}.$$

Since the arguments of L -s are local q -oscillator generators, $\mathbf{T}(u, v) \in \mathcal{H}^{\otimes NM}$ is by definition a layer-to-layer transfer matrix, its graphical representation is again the classical Fig. 3. The model is integrable since

$$(61) \quad \mathbf{T}(u, v) \mathbf{T}(u', v') = \mathbf{T}(u', v') \mathbf{T}(u, v),$$

i.e. the coefficients $\mathbf{t}_{n,m}$ in (60) form the set of the integrals of motion.

Now we give the technical proof of the commutativity (61). The commutativity of layer-to-layer transfer matrices follow from a proper tetrahedron equation [4]. In addition to $L_{\alpha,\beta}[\mathcal{A}_v]$ (54), define

$$(62) \quad \tilde{L}_{\alpha,\beta}[\mathcal{A}_0] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_0(-q)^{\mathbf{h}_0} & \mathbf{y}_0 & 0 \\ 0 & q^{-1}\lambda_0\mu_0\mathbf{x}_0 & \mu_0(-q)^{\mathbf{h}_0} & 0 \\ 0 & 0 & 0 & q^{-1}\lambda_0\mu_0 \end{pmatrix},$$

where $\mathcal{A}_0 \sim (\mathcal{H}_0; \lambda_0, \mu_0)$. The *constant* tetrahedron equation for L and \tilde{L} ,

$$(63) \quad \tilde{L}_{\alpha,\alpha'}[\mathcal{A}_0]\tilde{L}_{\beta,\beta'}[\mathcal{A}_0]L_{\alpha,\beta}[\mathcal{A}]L_{\alpha',\beta'}[\mathcal{A}] = L_{\alpha',\beta'}[\mathcal{A}]L_{\alpha,\alpha'}[\mathcal{H}]\tilde{L}_{\beta,\beta'}[\mathcal{A}_0]\tilde{L}_{\alpha,\alpha'}[\mathcal{A}_0]$$

may be verified directly, it is just 16×16 matrix equation with the operator-valued entries.

Another technical relation is

$$(64) \quad D_\alpha(u)D_{\alpha'}(u')\tilde{L}_{\alpha,\alpha'}[\mathcal{A}_0] = \left(\frac{u}{u'}\right)^{\mathbf{h}_0}\tilde{L}_{\alpha,\alpha'}[\mathcal{A}_0]\left(\frac{u'}{u}\right)^{\mathbf{h}_0}D_\alpha(u)D_{\alpha'}(u'),$$

where D is given by (42). Combining (63) for the whole lattice, we come to

$$(65) \quad \tilde{\mathbb{L}}_{\alpha,\alpha'}\tilde{\mathbb{L}}_{\beta,\beta'}\mathbb{L}_{\alpha,\beta}\mathbb{L}_{\alpha',\beta'} = \mathbb{L}_{\alpha',\beta'}\mathbb{L}_{\alpha,\beta}\tilde{\mathbb{L}}_{\beta,\beta'}\tilde{\mathbb{L}}_{\alpha,\alpha'},$$

where besides the “monodromies” (40) of $L[\mathcal{A}_v]$

$$(66) \quad \mathbb{L}_{\alpha,\beta} = \prod_{n,m}^{\sim} L_{\alpha_n,\beta_m}[\mathcal{A}_{n,m}] \text{ and } \mathbb{L}_{\alpha',\beta'} = \prod_{n,m}^{\sim} L_{\alpha'_n,\beta'_m}[\mathcal{A}_{n,m}],$$

we used

$$(67) \quad \tilde{\mathbb{L}}_{\alpha,\alpha'} = \prod_n^{\sim} \tilde{L}_{\alpha_n,\alpha'_n}[\mathcal{A}_0] \text{ and } \tilde{\mathbb{L}}_{\beta,\beta'} = \prod_m^{\sim} \tilde{L}_{\beta_m,\beta'_m}[\mathcal{A}_0].$$

Multiplying (65) by $D_\alpha(u)D_\beta(v)D_{\alpha'}(u')D_{\beta'}(v')$ and taking into account (64), we get

$$(68) \quad \tilde{\mathbb{L}}_{\alpha,\alpha'}\left(\frac{u'v}{uv'}\right)^{\mathbf{h}_0}\tilde{\mathbb{L}}_{\beta,\beta'} \cdot D_\alpha(u)D_\beta(v)\mathbb{L}_{\alpha,\beta} \cdot D_{\alpha'}(u')D_{\beta'}(v')\mathbb{L}_{\alpha',\beta'} = \\ D_{\alpha'}(u')D_{\beta'}(v')\mathbb{L}_{\alpha',\beta'} \cdot D_\alpha(u)D_\beta(v)\mathbb{L}_{\alpha,\beta} \cdot \left(\frac{u'}{u}\right)^{\mathbf{h}_0}\tilde{\mathbb{L}}_{\beta,\beta'}\tilde{\mathbb{L}}_{\alpha,\alpha'}\left(\frac{v}{v'}\right)^{\mathbf{h}_0}.$$

Now taking the trace over the representation space \mathcal{F}_0 of \mathcal{H}_0 and denoting

$$(69) \quad \mathbb{M}_{\alpha,\beta,\alpha',\beta'} = \text{Trace}_{\mathcal{F}_0}\left(\tilde{\mathbb{L}}_{\alpha,\alpha'}\left(\frac{u'v}{uv'}\right)^{\mathbf{h}_0}\tilde{\mathbb{L}}_{\beta,\beta'}\right),$$

we come to the final similarity relation

$$\mathbb{M}_{\alpha,\beta,\alpha',\beta'} \cdot D_{\alpha}(u)D_{\beta}(v)\mathbb{L}_{\alpha,\beta} \cdot D_{\alpha'}(u')D_{\beta'}(v')\mathbb{L}_{\alpha',\beta'} =$$

(70)

$$D_{\alpha'}(u')D_{\beta'}(v')\mathbb{L}_{\alpha',\beta'} \cdot D_{\alpha}(u)D_{\beta}(v)\mathbb{L}_{\alpha,\beta} \cdot \mathbb{M}_{\alpha,\beta,\alpha',\beta'} ,$$

and therefore two transfer matrices $\mathbf{T}(u, v) = \text{Trace}_{\alpha,\beta} \left(D_{\alpha}(u)D_{\beta}(v)\mathbb{L}_{\alpha,\beta} \right)$ commute.

In the limit $q \rightarrow 1$ coefficients of (60) become the involutive moduli of the spectral curve (28). Since the moduli of the classical spectral curve are independent and the spectral curve defines completely the solution of the classical model [3], the set of integrals of motion $\mathbf{t}_{n,m}$ is complete.

The combinatorial representation for $\mathbf{T}(u, v)$ (59) is equivalent to the combinatorial representation for $T(u, v)$ (43). The vertex factors of $L[A_v]$ (37) in Fig. 5 are to be replaced by corresponding elements of $L[\mathcal{A}_v]$ (54). Equations (45) and (47) remain unchanged. In particular, with the help of the combinatorial representation one may easily see

$$(71) \quad \mathbf{T}(u, 0) = \prod_n \left(1 + u \prod_m \lambda_{n,m} q^{\mathbf{h}_{n,m}} \right) , \quad \mathbf{T}(0, v) = \prod_m \left(1 + v \prod_n \mu_{n,m} q^{\mathbf{h}_{n,m}} \right) ,$$

so that the basic integrals of motion are $q^{\mathcal{J}_n}$ and $q^{\mathcal{K}_m}$,

$$(72) \quad \mathcal{J}_n = \sum_m \mathbf{h}_{n,m} \text{ and } \mathcal{K}_m = \sum_n \mathbf{h}_{n,m} .$$

Eigenvalues of \mathcal{J}_n and \mathcal{K}_m fix a subspace in the state space of the model.

2.3. Transfer matrix (59) and 2d quantum inverse scattering method. The transfer matrix (59) may be identically rewritten as the trace of 2d monodromy matrix

$$(73) \quad \mathbf{T}(u, v) = \text{Trace}_{\beta} \left(D_{\beta}(v) \prod_n^{\widehat{}} \mathcal{L}_{\beta}^{(n)}(u) \right) ,$$

where

$$(74) \quad \mathcal{L}_{\beta}^{(n)}(u) = \text{Trace}_{\alpha_n} \left(D_{\alpha_n}(u) \prod_m^{\widehat{}} L_{\alpha_n, \beta_m}[\mathcal{A}_{n,m}] \right) .$$

Equations (63) and (64) provide in particular

$$(75) \quad \widetilde{L}_{\alpha, \alpha'}^{(0)} \cdot \left(\frac{u'}{u} \right)^{\mathbf{h}_0} \widetilde{L}_{\beta_m, \beta'_m}^{(0)} \cdot D_{\alpha}(u) L_{\alpha, \beta_m}^{(m)} \cdot D_{\alpha'}(u') L_{\alpha', \beta'_m}^{(m)} =$$

$$D_{\alpha'}(u') L_{\alpha', \beta'_m}^{(m)} \cdot D_{\alpha}(u) L_{\alpha, \beta_m}^{(m)} \cdot \left(\frac{u'}{u} \right)^{\mathbf{h}_0} \widetilde{L}_{\beta_m, \beta'_m}^{(0)} \cdot \widetilde{L}_{\alpha, \alpha'}^{(0)} ,$$

and therefore

$$(76) \quad \mathfrak{R}_{\beta,\beta'}(u/u') \mathfrak{L}_{\beta}(u) \mathfrak{L}_{\beta'}(u') = \mathfrak{L}_{\beta'}(u') \mathfrak{L}_{\beta}(u) \mathfrak{R}_{\beta,\beta'}(u/u') ,$$

where

$$(77) \quad \mathfrak{R}_{\beta,\beta'}(u/u') = \text{Trace}_{\mathcal{F}_0} \left\{ \left(\frac{u'}{u} \right)^{\mathbf{h}_0} \prod_m^{\sim} \tilde{L}_{\beta_m,\beta'_m}[\mathcal{A}_0] \right\} .$$

In the same way, the other direction of the lattice may be chosen, and the dual Lax operator

$$(78) \quad \mathfrak{L}_{\alpha}^{(m)}(v) = \text{Trace}_{\beta_m} \left(D_{\beta_m}(v) \prod_n^{\sim} L_{\alpha_n,\beta_m}[\mathcal{A}_{n,m}] \right)$$

may be considered.

Further in this subsection we recall the structure of $2d$ Lax operator (74) and its R -matrix (77). Matrix $\mathfrak{L}_{\beta}^{(n)}$, as the matrix in the auxiliary space $V_{\beta} = (\mathbb{C}^2)^{\otimes M}$ with operator-valued matrix elements from $\mathcal{H}^{\otimes M}$, has a block-diagonal structure. Let, in the concordance with (34) and (36), $|0\rangle$ and $|1\rangle$ be the basis of \mathbb{C}^2 . Define $\varphi_0, \varphi_j, \varphi_{j,j'}$, etc. as the following elements of V_{β} :

$$(79) \quad \begin{aligned} \varphi_0 &= |0\rangle \otimes \cdots \otimes |0\rangle , \\ \varphi_j &= |0\rangle \otimes \cdots \underset{j^{\text{th}} \text{ place}}{|1\rangle} \otimes \cdots \otimes |0\rangle , \end{aligned}$$

etc., in general φ_{j_1,\dots,j_m} has $|1\rangle$ on $j_1^{\text{th}}, \dots, j_m^{\text{th}}$ places, $j_1 < \dots < j_m$. In the basis of φ

$$(80) \quad \mathfrak{L}_{\beta}^{(n)}(u) \varphi_0 = \varphi_0 L_0^{(n)}(u) , \quad \text{where} \quad L_0(u) = 1 + u \prod_m \lambda_{n,m} q^{\mathbf{h}_{n,m}} .$$

Next

$$(81) \quad \mathfrak{L}_{\beta}^{(n)}(u) \varphi_k = \sum_{j=1}^M \varphi_j L_{j,k}^{(n)}(u) ,$$

where $L_{j,k}(u)$ are matrix elements of Lax operator for the vector representation of $\mathcal{U}_q(\widehat{sl}_M)$, they are given in [1]. In general,

$$(82) \quad \mathfrak{L}_{\beta}(u) = \bigoplus_{m=0}^M L_{\omega_m}(u) ,$$

where block ω_m corresponds to $\frac{M!}{m!(M-m)!}$ -dimensional vector space $\pi_{\omega_m} \in V_{\beta}$ with the basis φ_{j_1,\dots,j_m} , $j_1 < \dots < j_m$.

Matrix $\mathfrak{R}_{\beta, \beta'}$ (77) has the block structure as well,

$$(83) \quad \mathfrak{R}_{\beta, \beta'}(u) = \bigoplus_{m, m'=0}^M \lambda_0^m \mu_0^{m'} R_{\omega_m, \omega_{m'}}(u),$$

where λ_0, μ_0 are extra parameters of \mathcal{A}_0 (77), and $R_{\omega_m, \omega_{m'}}$ is the $\mathcal{U}_q(\widehat{sl}_M)$ R -matrix for the representations $\pi_{\omega_m} \otimes \pi_{\omega_{m'}}$. In particular, in the sector $\pi_{\omega_1} \otimes \pi_{\omega_1}$ with the basis φ_j (79), one can obtain the fundamental R -matrix (we used the Fock space (51) for \mathcal{H}_0 for the calculation of the trace in (77)):

$$(84) \quad \mathfrak{R}_{\beta, \beta'}(u) \varphi_j \otimes \varphi_k = \lambda_0 \mu_0 \frac{u}{(u - q^2)(u - 1)} \sum_{j', k'} \varphi_{j'} \otimes \varphi_{k'} R_{j', k'}^{j, k}(u),$$

where

$$(85) \quad \begin{aligned} R_{j, k}^{j, k}(u) &= u - 1, \quad R_{j, j}^{j, j} = q^{-1}(u - q^2), \\ R_{k, j}^{j, k}(u) &= q^{-1}(1 - q^2) \text{ for } j < k \text{ and } R_{k, j}^{j, k}(u) = q^{-1}u(1 - q^2) \text{ for } j > k. \end{aligned}$$

Lax operator $\mathfrak{L}_{\beta}^{(n)}(u)$ (74) has the center $q^{\mathcal{J}_n}$ (72), while $q^{\mathcal{K}_m}$ is the center of $\mathfrak{L}_{\alpha}^{(m)}(v)$ (78). The quantum space of \mathfrak{L}_{β} is $\mathcal{F}^{\otimes M}$, it may be decomposed as

$$(86) \quad \mathcal{F}^{\otimes M} = \bigoplus_{J=0}^{\infty} \pi_{J\omega_1},$$

where J is the eigenvalue of \mathcal{J} , and $\pi_{J\omega_1}$ is the rank- J symmetrical tensor representation of $\mathcal{U}_q(\widehat{sl}_M)$ (dominant weight $J\omega_1$).

We would like to conclude this subsection by the example of $M = 2$ containing the six-vertex model. The block-diagonal structure of \mathfrak{L} (74) is

$$(87) \quad \mathfrak{L}(u) = \begin{pmatrix} 1 + u\lambda_1\lambda_2q^{\mathbf{h}_1+\mathbf{h}_2} & 0 & 0 \\ 0 & L(u) & 0 \\ 0 & 0 & \mu_1\mu_2(q^{\mathbf{h}_1+\mathbf{h}_2} + q^{-2}u\lambda_1\lambda_2) \end{pmatrix},$$

where index n of (74) is omitted, and 2×2 central block is

$$(88) \quad L(u) = \begin{pmatrix} \mu_1(q^{\mathbf{h}_1} - u\lambda_1\lambda_2q^{\mathbf{h}_2-1}) & -q^{-1}u\lambda_1\mu_1\mathbf{x}_1\mathbf{y}_2 \\ -q^{-1}\mathbf{y}_1\lambda_2\mu_2\mathbf{x}_2 & \mu_2(q^{\mathbf{h}_2} - u\lambda_1\lambda_2q^{\mathbf{h}_1-1}) \end{pmatrix}.$$

Fixed integer $\mathcal{J} = \mathbf{h}_1 + \mathbf{h}_2$ in the quantum space corresponds to spin $\mathcal{J}/2$ representation of sl_2 . For spin $1/2$ representation ($\mathcal{J} = 1$), matrix elements of (88) may be presented by

$$(89) \quad q^{\mathbf{h}_1} = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad q^{\mathbf{h}_2} = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, \quad \mathbf{x}_1\mathbf{y}_2 = \begin{pmatrix} 0 & 0 \\ 1 - q^2 & 0 \end{pmatrix}, \quad \mathbf{y}_1\mathbf{x}_2 = \begin{pmatrix} 0 & 1 - q^2 \\ 0 & 0 \end{pmatrix},$$

and in the homogeneous case $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$ the matrix (88) becomes exactly the six-vertex R -matrix.

Let the chain of Lax matrices (74) is given, $n = 1, 2, \dots, N$. Six vertex model corresponds to the choice $\mathcal{J}_n = 1$ for all n . The values of two extra integrals of motion, \mathcal{K}_1 and \mathcal{K}_2 of (72), are related to total spin of the chain: \mathcal{K}_1 is the number of “spins up”, \mathcal{K}_2 is the number of “spins down”.

3. QUANTUM CURVE

Solution of the classical equations of motion is based on the notion of the spectral curve $J(u, v) = 0$ (28). In the previous section we succeed in construction of the “quantum partition function” $\mathbf{T}(u, v)$ producing the set of the integrals of motion, but we did not answer the question: what is the quantum analogue of $J(u, v) = 0$?

The answer is the following (we are repeating the Introduction). Let \mathbf{u}, \mathbf{v} be an additional auxiliary Weyl pair,

$$(90) \quad \mathcal{W} : \quad \mathbf{u} \mathbf{v} = q^2 \mathbf{v} \mathbf{u}$$

serving two variants of “quantum-mechanical” notations

$$(91) \quad \begin{aligned} \langle Q|u\rangle &= Q(u), \quad \mathbf{u}|u\rangle = |u\rangle u, \quad \mathbf{v}|u\rangle = |q^2 u\rangle v, \\ \text{or} \\ \langle v|\overline{Q}\rangle &= \overline{Q}(v), \quad \langle v|\mathbf{v} = v\langle v|, \quad \langle v|\mathbf{u} = u\langle q^2 v|. \end{aligned}$$

For the given layer-to-layer transfer matrix $\mathbf{T}(u, v)$ (59,60), define

$$(92) \quad \mathbf{J}(\mathbf{u}, \mathbf{v}) = \sum_{n=0}^N \sum_{m=0}^M (-q)^{-nm} (-\mathbf{u})^n (-\mathbf{v})^m \mathbf{t}_{n,m}.$$

One may easily see, in the limit $q \rightarrow 1$ (92) becomes (48). Operator $\mathbf{J}(\mathbf{u}, \mathbf{v})$ belongs to $\mathcal{H}^{\otimes NM} \otimes \mathcal{W}$. Let $|t\rangle \in \mathcal{F}^{\otimes NM}$ be an eigenvector of $\mathbf{T}(u, v)$,

$$(93) \quad \mathbf{t}_{n,n} |t\rangle = |t\rangle t_{n,n},$$

so that

$$(94) \quad \mathbf{J}(\mathbf{u}, \mathbf{v}) \left(|t\rangle \otimes id \right) = \left(|t\rangle \otimes id \right) J(\mathbf{u}, \mathbf{v}),$$

where $J(\mathbf{u}, \mathbf{v}) \in \mathcal{W}$, (11). Define a linear space Ψ by

$$(95) \quad \mathbf{J}(\mathbf{u}, \mathbf{v}) |\Psi\rangle = 0 \quad \text{or} \quad \langle \Psi | \mathbf{J}(\mathbf{u}, \mathbf{v}) = 0.$$

The linear space may be decomposed with respect to the basis $|t\rangle$,

$$(96) \quad |\Psi\rangle = \bigoplus_t |t\rangle \otimes |\overline{Q}_t\rangle \quad \text{or} \quad \langle\Psi| = \bigoplus_t \langle t| \otimes \langle Q_t|.$$

The point is that $\langle Q_t|$ and $|\overline{Q}_t\rangle$ have a simple and predictable structure, in particular $\langle Q_t|u\rangle$ and $\langle v|\overline{Q}_t\rangle$ may be defined as polynomials.

Functions $\langle Q|u\rangle$ and $\langle v|\overline{Q}\rangle$ (index t is usually omitted) obey the linear difference equations (95),

$$(97) \quad \langle\Psi| \mathbf{J}(\mathbf{u}, \mathbf{v}) |t\rangle \otimes |u\rangle = 0 \quad \text{and} \quad \langle t| \otimes \langle v| \mathbf{J}(\mathbf{u}, \mathbf{v}) |\Psi\rangle = 0,$$

which are exactly Eqs. (7) and (8).

In this section, we will derive (92) and (95) as a solution of quantized linear problem (15) for the whole auxiliary lattice. A non-trivial problem is how to define a quantized linear problem in such a way, that each linear variable belongs to the same subspace (subspace (95) in the final solution). This is the problem of consistency of linear problem and invariance of quantum curve.

3.1. Generalized matrix of coefficients of linear problem. Consider the linear problem (15) on an arbitrary lattice with u, v -periodical boundary conditions. The set of local linear problems

$$(98) \quad \begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \end{pmatrix} = X_v \cdot \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix}, \quad X_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$$

may be rewritten for the whole lattice in a matrix form,

$$(99) \quad \sum_j \ell_{k,j} \psi_j = 0.$$

The fragments in (99), corresponding to (98), are

$$(100) \quad \ell = \begin{pmatrix} \ddots & \vdots & \vdots & 0 & 0 & \ddots \\ 0 & 1 & 0 & -a_v & -b_v & 0 \\ 0 & 0 & 1 & -c_v & -d_v & 0 \\ \dots & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ \ddots & \vdots & \vdots & 0 & 0 & \ddots \end{pmatrix} \quad \text{for} \quad \psi = \begin{pmatrix} \vdots \\ \psi'_\alpha \\ \psi'_\beta \\ \psi_\alpha \\ \psi_\beta \\ \vdots \end{pmatrix}.$$

In the quantum world, matrix elements $\ell_{k,j}$ are non-commutative operators $\ell_{k,j}$. Therefore (99) may have two slightly different forms,

$$(101) \quad \text{A). } \sum_j \ell_{k,j} |\psi_j\rangle = 0 \quad \text{or} \quad \text{B). } \sum_j \langle \psi_j | \ell_{k,j} = 0 .$$

Let the operators $\ell_{k,j}$ obey the following exchange relations:

$$(102) \quad \begin{aligned} & \ell_{k,j} \ell_{k',j'} - \ell_{k',j} \ell_{k,j'} = \ell_{k',j'} \ell_{k,j} - \ell_{k,j'} \ell_{k',j} , \quad k \neq k' , \quad j \neq j' , \\ & \text{and} \\ & \ell_{k,j} \ell_{k',j} = \ell_{k',j} \ell_{k,j} . \end{aligned}$$

The aim of this subsection is to establish fundamental properties of such ℓ and the form of solution of (101) A) and B).

Define

$$(103) \quad \det \ell \stackrel{\text{def}}{=} \sum_{\sigma} (-)^{\sigma} \prod_j^{\sim} \ell_{\sigma_j, j} \equiv \sum_{\sigma} (-)^{\sigma} \ell_{\sigma_1, 1} \ell_{\sigma_2, 2} \ell_{\sigma_3, 3} \dots$$

where σ are the permutations of the indices $1, 2, 3, \dots$. According to the first relation of (102), the definition (103) is invariant with respect to the ordering of j , for instance

$$(104) \quad \sum_{\sigma} (-)^{\sigma} \ell_{\sigma_1, 1} \ell_{\sigma_2, 2} \ell_{\sigma_3, 3} \dots = \sum_{\sigma} (-)^{\sigma} \ell_{\sigma_2, 2} \ell_{\sigma_1, 1} \ell_{\sigma_3, 3} \dots = \dots ,$$

and in general

$$(105) \quad \det \ell = \sum_{\sigma} (-)^{\sigma} \prod_j^{\sim} \ell_{\sigma_{\tau_j}, \tau_j}$$

where τ is any permutation of $1, 2, 3, \dots$. Define next the algebraic supplements $\mathbf{A}_{j,k}$ of ℓ (adjoint matrix) as (103)-determinants of the minors of ℓ ,

$$(106) \quad \sum_k \mathbf{A}_{j,k} \ell_{k,j} = \sum_k \ell_{k,j} \mathbf{A}_{j,k} = \det \ell .$$

The first equality here follows from (105), the last one in the definition of $\mathbf{A}_{j,k}$. The second line of (102) provides

$$(107) \quad \sum_k \mathbf{A}_{j,k} \ell_{k,j'} = \sum_k \ell_{k,j'} \mathbf{A}_{j,k} = 0 \quad \text{if } j \neq j' .$$

Therefore $(\det \ell)^{-1} \mathbf{A}_{j,k}$ and $\mathbf{A}_{j,k} (\det \ell)^{-1}$ are two variants of inverse matrices,

$$(108) \quad \sum_k (\det \ell)^{-1} \mathbf{A}_{j,k} \ell_{k,j'} = \sum_k \ell_{k,j'} \mathbf{A}_{j,k} (\det \ell)^{-1} = \delta_{j,j'} ,$$

or, since the inverse matrix must be both left- and right-inverse,

$$(109) \quad \sum_j \ell_{k,j} (\det \ell)^{-1} \mathbf{A}_{j,k_0} = \sum_j \mathbf{A}_{j,k_0} (\det \ell)^{-1} \ell_{k,j} = \delta_{k,k_0} .$$

The main property of the elements of inverse matrices is the commutativity of their matrix elements with the same k_0 , e.g. for the variant A),

$$(110) \quad [(\det \ell)^{-1} \mathbf{A}_{j,k_0} , (\det \ell)^{-1} \mathbf{A}_{j',k_0}] = 0 .$$

It is a particular case of the following statement:

$$(111) \quad \sum_j \ell_{k,j} \mathbf{m}_j = \varepsilon_k , \quad \varepsilon_k \in \mathbb{C} \quad \Rightarrow \quad [\mathbf{m}_j, \mathbf{m}_{j'}] = 0 .$$

To prove (111), consider

$$(112) \quad c_{k,k'} = \sum_{j < j'} (\ell_{k,j} \ell_{k',j'} - \ell_{k',j} \ell_{k,j'}) (\mathbf{m}_j \mathbf{m}_{j'} - \mathbf{m}_{j'} \mathbf{m}_j) , \quad k < k' .$$

Due to the first relation of (102) and definition (111), $c_{k,k'} = \varepsilon_k \varepsilon_{k'} - \varepsilon_{k'} \varepsilon_k = 0$. From the other side, let $\mathbf{A}_{j,j'|k,k'}^{(2)}$ is the matrix of the second algebraical supplements of ℓ :

$$(113) \quad \sum_{k < k'} \mathbf{A}_{i,i'|k,k'}^{(2)} (\ell_{k,j} \ell_{k',j'} - \ell_{k',j} \ell_{k,j'}) = \delta_{i,j} \delta_{i',j'} \det \ell , \quad i < i' , \quad j < j' .$$

Then $0 = \sum_{k < k'} \mathbf{A}_{i,i'|k,k'}^{(2)} c_{k,k'} = \det \ell \cdot (\mathbf{m}_j \mathbf{m}_{j'} - \mathbf{m}_{j'} \mathbf{m}_j)$, what proves the commutativity of \mathbf{m}_j . The commutativity (110) corresponds to $\varepsilon_k = \delta_{k,k_0}$.

For $\varepsilon_k = \delta_{k,k_0}$ with fixed k_0 , let

$$(114) \quad \mathbf{m}_{j,j_0} = (\mathbf{m}_{j_0})^{-1} \mathbf{m}_j = \mathbf{A}_{j_0,k_0}^{-1} \mathbf{A}_{j,k_0} , \quad \mathbf{m}'_{j,j_0} = \mathbf{A}_{j,k_0} \mathbf{A}_{j_0,k_0}^{-1} .$$

Then the set \mathbf{m}_{j,j_0} is commutative (k_0 is fixed), as well the set \mathbf{m}'_{j,j_0} , and besides

$$(115) \quad \det \ell \cdot \mathbf{m}_{j,j_0} = \mathbf{m}'_{j,j_0} \cdot \det \ell .$$

Inverse relations (109) give

$$(116) \quad \sum_j \ell_{k,j} \mathbf{m}_{j,j_0} = \delta_{k,k_0} \mathbf{A}_{j_0,k_0}^{-1} \det \ell , \quad \sum_j \mathbf{m}'_{j,j_0} \ell_{k,j} = \delta_{k,k_0} \det \ell \mathbf{A}_{j_0,k_0}^{-1} .$$

Equations (116) give the formal solution of the linear problem (101) A) and B): let $|\psi_{j_0}\rangle$ or $\langle\psi_{j_0}|$ for some j_0 are defined by $\det \ell \cdot |\psi_{j_0}\rangle = 0$ or $\langle\psi_{j_0}| \cdot \det \ell = 0$. Then

$$(117) \quad \text{A). } |\psi_j\rangle = \mathbf{m}_{j,j_0} |\psi_{j_0}\rangle \quad \text{or} \quad \text{B). } \langle\psi_j| = \langle\psi_{j_0}| \mathbf{m}'_{j,j_0} .$$

Common feature of all $|\psi_j\rangle$ or $\langle\psi_j|$ is

$$(118) \quad \text{A). } \det \ell \cdot |\psi_j\rangle = 0 \quad \text{or} \quad \text{B). } \langle\psi_j| \cdot \det \ell = 0 \quad \forall j .$$

3.2. Extended algebra of observables. To establish the relation between the algebra of matrix elements (102) and our case of X (53), we have to introduce the extended algebra of observables $\mathfrak{A}_v = \mathcal{H}_v \otimes \mathcal{W}_v$. Define

$$(119) \quad X[\mathfrak{A}_v] = \begin{pmatrix} \lambda_v q^{\mathbf{h}_v} & \nu_v \mathbf{y}_v \\ \nu_v \mathbf{x}_v & \mu_v q^{\mathbf{h}_v} \end{pmatrix} ,$$

where $q^{\mathbf{h}_v}, \mathbf{x}_v, \mathbf{y}_v$ are generators of q -oscillator \mathcal{H}_v (1) and λ_v, μ_v, ν_v are the generators of Weyl algebra \mathcal{W}_v :

$$(120) \quad \lambda_v \mu_v = q^2 \mu_v \lambda_v , \quad \nu_v^2 = -\mu_v \lambda_v .$$

The q -oscillators and the Weyl algebra elements for different vertices always commute.

Matrix ℓ , defined according to (100) for vertex matrix $X[\mathfrak{A}_v]$ (119), belongs exactly to the class (102) since

$$(121) \quad \begin{aligned} \nu \mathbf{x} \nu \mathbf{y} - \lambda q^{\mathbf{h}} \mu q^{\mathbf{h}} &= \nu \mathbf{y} \nu \mathbf{x} - \mu q^{\mathbf{h}} \lambda q^{\mathbf{h}} , \\ \text{and} \\ [\nu \mathbf{y} , \mu q^{\mathbf{h}}] &= [\nu \mathbf{x} , \lambda q^{\mathbf{h}}] = 0 \end{aligned}$$

and elements of \mathfrak{A}_v for different vertices commute. Determinant of ℓ has a combinatorial representation of Fig. 5, eqs. (45,47,48) with

$$(122) \quad L_{\alpha,\beta}[\mathfrak{A}_v] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_v q^{\mathbf{h}_v} & \nu_v \mathbf{y}_v & 0 \\ 0 & \nu_v \mathbf{x}_v & \mu_v q^{\mathbf{h}_v} & 0 \\ 0 & 0 & 0 & \nu_v^2 \end{pmatrix} .$$

Let

$$(123) \quad \det \ell \stackrel{\text{def}}{=} \mathbf{J}[\mathfrak{A}] = \sum (-)^{nm+n+m} u^n v^m \mathbf{J}_{n,m}[\mathfrak{A}] ,$$

where u and v are \mathbb{C} -valued spectral parameters introduced according to (22). Let locally

$$(124) \quad \lambda_v = \lambda_v e^{Q_v} , \quad \mu_v = \mu_v e^{P_v} \quad \text{and therefore} \quad \nu_v^2 = -q^{-1} \lambda_v \mu_v e^{P_v+Q_v} , \quad [Q_v, P_v] = \log q^2 .$$

In the combinatorial representation of the determinant (Fig. 5, eqs. (45,47,48)), consider a path $C_{n,m}$ of the homotopy class $n\mathcal{A} + m\mathcal{B}$. A monomial summand, corresponding to this path, may be factorized as

$$(125) \quad \mathbf{J}_{C_{n,m}} = \mathbf{t}_{C_{n,m}} e^{\phi(C_{n,m})}.$$

Monomial $\mathbf{t}_{C_{n,m}}$ gathers all the q -oscillators and \mathbb{C} -valued parameters λ_v, μ_v and $-q^{-1}\lambda_v\mu_v$, and therefore it is exactly the $C_{n,m}$ -monomial of $\mathbf{T}(u, v)$ (up to unessential renormalization of \mathbf{x}_v and \mathbf{y}_v). Operator ϕ is a sum of local Q_v and P_v . One may easily see,

$$(126) \quad [\phi(C_{n,m}), \phi(C'_{n',m'})] = (nm' - mn') \log q^2 \quad \forall C, C'.$$

Let $C_{1,0}$ and $C_{0,1}$ be two particular fixed paths. Due to (126), all

$$(127) \quad \tilde{\phi}(C_{n,m}) = \phi(C_{n,m}) - n\phi(C_{1,0}) - m\phi(C_{0,1})$$

commute. Therefore one may diagonalize them simultaneously and without loss of generality (since λ_v, μ_v are free) put

$$(128) \quad \tilde{\phi}(C_{n,m}) \equiv 0.$$

On this condition,

$$(129) \quad \phi(C_{n,m}) \rightarrow n\phi(C_{1,0}) + m\phi(C_{0,1}) \equiv nQ_0 + mP_0,$$

and exponent of $\phi(C_{n,m})$, together with the combinatorial factor $(-)^{nm+n+m}u^n v^m$, becomes

$$(130) \quad (-)^{nm+n+m} e^{nQ_0+mP_0} u^n v^m = (-q)^{-nm} (-\mathbf{u})^n (-\mathbf{v})^m, \quad \mathbf{u} = e^{Q_0} u, \quad \mathbf{v} = e^{P_0} v.$$

Therefore, on the subspace (128), $\mathbf{J}[\mathfrak{A}]$ becomes exactly $\mathbf{J}(\mathbf{u}, \mathbf{v})$ (92).

Definition of \mathbf{u}, \mathbf{v} (130) on the subspace (128) may be written in the terms of λ_v, μ_v as

$$(131) \quad u \prod_m \lambda_{nm} \xrightarrow{\tilde{\phi}=0} \mathbf{u} \prod_m \lambda_{nm}, \quad v \prod_n \mu_{nm} \xrightarrow{\tilde{\phi}=0} \mathbf{v} \prod_n \mu_{nm}.$$

3.3. Structure of a solution of linear problem. Turn now to the structure of linear spaces $|\Psi\rangle$ and $\langle\Psi|$ defined by

$$(132) \quad \text{A). } \mathbf{J}[\mathfrak{A}] |\Psi\rangle = 0 \quad \text{and} \quad \text{B). } \langle\Psi| \mathbf{J}[\mathfrak{A}] = 0 .$$

According to (110,114,115), each of $|\psi_j\rangle$ and $\langle\psi_j|$ belong to these subspaces, and there exist sets of commutative operators \mathbf{m}_j and \mathbf{m}'_j (117) such that $|\psi_j\rangle = \mathbf{m}_j |\Psi\rangle$ and $\langle\psi_j| = \langle\Psi| \mathbf{m}'_j$. In particular, one may consider the linear spaces $|\Psi\rangle$ and $\langle\Psi|$ in the basis of diagonal $\mathbf{m}_j, \mathbf{m}'_j$:

$$(133) \quad \text{A). } |\psi_j\rangle = |\Psi\rangle m_j, \quad \text{or} \quad \text{B). } \langle\psi_j| = m_j \langle\Psi| ,$$

where m_j are eigenvalues of \mathbf{m}_j etc. The every local pair of linear equations (98) take the form

$$(134) \quad \text{A). } \begin{cases} (1 - \lambda_v q^{\mathbf{h}_v} A_v - \nu_v \mathbf{y}_v B_v) |\Psi\rangle = 0 \\ (1 - \nu_v \mathbf{x}_v C_v - \mu_v q^{\mathbf{h}_v} D_v) |\Psi\rangle = 0 \end{cases} \quad \text{or} \quad \text{B). } \begin{cases} \langle\Psi| (1 - A_v \lambda_v q^{\mathbf{h}_v} - B_v \nu_v \mathbf{y}_v) = 0 \\ \langle\Psi| (1 - C_v \nu_v \mathbf{x}_v - D_v \mu_v q^{\mathbf{h}_v}) = 0 \end{cases}$$

where in the notations of (98) $A_v = \frac{m_\alpha}{m_{\alpha'}}$, $B_v = \frac{m_\beta}{m_{\beta'}}$, $C_v = \frac{m_\alpha}{m_{\beta'}}$ and $D_v = \frac{m_\beta}{m_{\alpha'}}$.

The linear spaces $|\Psi\rangle$ and $\langle\Psi|$ belong to a module of $\mathcal{H}^{\otimes NM} \otimes \mathcal{W}^{\otimes NM}$. Its “physical” part is $\mathcal{F}^{\otimes NM}$, the artificial Weyl part may be defined in many inequivalent ways. Let $|n\rangle$ be the state of $\mathcal{F}^{\otimes NM}$ with n_v bosons in the vertex \mathbf{v} , $n = \{n_v\}$. We will focus on the formal states of $\mathcal{W}^{\otimes NM}$ for (132) in the basis (133):

$$(135) \quad \text{A). } |\Psi_n\rangle = \left(\langle n| \otimes id \right) |\Psi\rangle \quad \text{or} \quad \langle\Psi_n| = \langle\Psi| \left(|n\rangle \otimes id \right) .$$

Equations (134) give for (135)

$$(136) \quad \begin{aligned} &\text{A). } \begin{cases} (1 - \lambda_v A_v) |\Psi_0\rangle = 0 \quad \forall \mathbf{v} \quad \text{and} \\ |\Psi_n\rangle = \prod_{\mathbf{v}} \frac{\nu_v^{-n_v} (\mu_v D_v; q^2)_n}{C_v^{n_v} \sqrt{(q^2; q^2)_{n_v}}} |\Psi_0\rangle \end{cases} \\ &\text{or} \\ &\text{B). } \begin{cases} \langle\Psi_0| (1 - \mu_v D_v) = 0 \quad \forall \mathbf{v} \quad \text{and} \\ \langle\Psi_n| = \langle\Psi_0| \prod_{\mathbf{v}} \frac{(\lambda_v A_v; q^2)_{n_v} \nu_v^{-n_v}}{B_v^{n_v} \sqrt{(q^2; q^2)_{n_v}}} \end{cases} \end{aligned}$$

Here we used the representation

$$(137) \quad \mathbf{x} |n\rangle = |n-1\rangle \sqrt{1-q^{2n}}, \quad \mathbf{y} |n\rangle = |n+1\rangle \sqrt{1-q^{2+2n}} .$$

Turn at the first to the conditions for Ψ_0 in (136). Since the total Fock vacuum is the eigenstate of q -oscillator counterpart of $\mathbf{J}[\mathfrak{A}]$, one may consider directly

$$(138) \quad \begin{aligned} \text{A). } 0 &= \left(\langle 0| \otimes id \right) \mathbf{J}[\mathfrak{A}] |\Psi\rangle = \mathbf{J}_0[\mathcal{W}^{\otimes NM}] |\Psi_0\rangle \\ \text{or} \\ \text{B). } 0 &= \langle \Psi| \mathbf{J}[\mathfrak{A}] \left(|0\rangle \otimes id \right) = \langle \Psi_0| \mathbf{J}_0[\mathcal{W}^{\otimes NM}] , \end{aligned}$$

where

$$(139) \quad \mathbf{J}_0[\mathcal{W}^{\otimes NM}] = \prod_m (1 - v \prod_n \mu_{nm}) \prod_n (1 - u \prod_m \lambda_{nm}) .$$

Evidently, \mathbf{J}_0 commutes with all $\tilde{\phi}$ (127) and therefore allows the projection $\mathcal{W}^{\otimes NM} \rightarrow \mathcal{W}$ (128). Simply applying (131),

$$(140) \quad \mathbf{J}_0[\mathcal{W}^{\otimes NM}] \xrightarrow{\tilde{\phi}=0} J_0(\mathbf{u}, \mathbf{v}) = \prod_m (1 - \mathbf{v} \prod_n \mu_{nm}) \prod_n (1 - \mathbf{u} \prod_m \lambda_{nm}) .$$

Now the state of the Weyl algebra \mathbf{u}, \mathbf{v} may be chosen in the most convenient way. For the case A the proper basis is $\langle v|$ of (91), and for the case B the proper basis is $|u\rangle$. If $u = u_n$ for the case A or $v = v_m$ for the case B,

$$(141) \quad u_n = \prod_m \lambda_{nm}^{-1} , \quad v_m = \prod_n \mu_{nm}^{-1} ,$$

then

$$(142) \quad \text{A). } \langle v|\Psi_0\rangle = 1 \quad \text{or} \quad \text{B). } \langle \Psi_0|u\rangle = 1 .$$

Let further $|t\rangle$ and $\langle t|$ be the eigenstates of \mathbf{t}_{nm} , (93). Analogously to (135) let

$$(143) \quad \text{A). } |\Psi_t\rangle = \left(\langle t| \otimes id \right) |\Psi\rangle \quad \text{or} \quad \text{B). } \langle \Psi_t| = \langle \Psi| \left(|t\rangle \otimes id \right) .$$

It follows from the second lines of (136)

$$(144) \quad \text{A). } |\Psi_t\rangle = \mathcal{P}_t(\boldsymbol{\nu}, \boldsymbol{\mu}) |\Psi_0\rangle \quad \text{or} \quad \text{B). } \langle \Psi_t| = \langle \Psi_0| \mathcal{P}'_t(\boldsymbol{\nu}, \boldsymbol{\lambda}) ,$$

where e.g. \mathcal{P}_t is a polynomial of $\boldsymbol{\mu}_{\mathbf{v}}$ of a total power not more that the number of bosons in $|t\rangle$, its structure with respect to $\boldsymbol{\nu}_{\mathbf{v}}$ is rather simple. In addition, since we consider the eigenstates in the Fock counterpart of $\mathfrak{A}^{\otimes NM}$, polynomials \mathcal{P}_t and \mathcal{P}'_t must commute with all $\tilde{\phi}$ (127) and therefore must allow the projection $\mathcal{W}^{\otimes NM} \rightarrow \mathcal{W}$ (128):

$$(145) \quad \mathcal{P}_t \rightarrow \mathbf{w}^{-J} P_t(\mathbf{v}) \quad \text{and} \quad \mathcal{P}'_t \rightarrow P'_t(\mathbf{u}) \mathbf{w}^{-K} ,$$

where $\mathbf{w}^2 = -\mathbf{v}\mathbf{u}$, \mathbf{w} -factor comes from $\nu_{\mathbf{v}}$ factors; J and K are integers; P_t and P'_t are polynomials of a power not higher than the total number of bosons in the state $|t\rangle$.

Taking now into account (142), we come to the final statement: in the restricted algebra $\mathcal{H}^{\otimes NM} \otimes \mathcal{W}$ the equations

$$(146) \quad \text{A). } \left(\langle t| \otimes \langle v| \right) \mathbf{J}(\mathbf{u}, \mathbf{v}) |\Psi\rangle = 0 \quad \text{and} \quad \text{B). } \langle \Psi | \mathbf{J}(\mathbf{u}, \mathbf{v}) \left(|t\rangle \otimes |u\rangle \right) = 0 .$$

have the solutions

$$(147) \quad \begin{aligned} &\text{A). if } u = u_n \text{ then } \langle v | \Psi \rangle = v^{-J_n/2} \overline{Q}_n(v) \\ &\text{and} \\ &\text{B). if } v = v_m \text{ then } \langle \Psi | u \rangle = u^{-K_m/2} Q_m(u) \end{aligned}$$

where u_n, v_m are given by (141), and the integers J_n, K_m and degrees of polynomials Q_n, \overline{Q}_m are not higher than the total number of bosons.

In the next section, considering the examples, we will see that J_n and K_m are the eigenvalues of \mathcal{J}_n and \mathcal{K}_m (72) and, moreover, the degrees of \overline{Q}_m and Q_n are exactly K_m and J_n .

In the next section we will add q^{-J_n} and q^{-K_m} to the definition of u_n and v_m . This allows one to cancel the half-integer pre-factors $v^{-J_n/2}$ of $\overline{Q}_n(v)$ and $u^{-K_m/2}$ of $Q_m(u)$. The corresponding values of u_n and v_m may be obtained via conditions $\overline{Q}(0) = 1$ and $Q_n(0) = 1$.

4. EXAMPLES

Let us illustrate (97) for the six-vertex model first, and then for an arbitrary square lattice.

4.1. Six-vertex chain. Consider the lattice with $M = 2$ and arbitrary N . This is the case of six-vertex model, where (97) becomes the Baxter equation. It follows from (87),

$$(148) \quad \begin{aligned} \sum_n u^n \mathbf{t}_{n,0} &= \prod_n (1 + u \lambda_{n,1} \lambda_{n,2} q^{\mathbf{h}_{n,1} + \mathbf{h}_{n,2}}) , \\ \sum_n u^n \mathbf{t}_{n,1} &= \mathbf{t}(u) , \\ \sum_n u^n \mathbf{t}_{n,2} &= \prod_n \mu_{n,1} \mu_{n,2} (q^{\mathbf{h}_{n,1} + \mathbf{h}_{n,2}} + q^{-2} u \lambda_{n,1} \lambda_{n,2}) , \end{aligned}$$

where $\mathbf{t}(u)$ is the transfer matrix for the Lax operator (88). Its opposite elements are

$$\mathbf{t}_{0,1} = \prod_n (\mu_{n,1} q^{\mathbf{h}_{n,1}}) + \prod_n (\mu_{n,2} q^{\mathbf{h}_{n,2}}) , \quad (149)$$

$$\mathbf{t}_{N,1} = \prod_n (-q^{-1} \lambda_{n,1} \lambda_{n,2} \mu_{n,2} q^{\mathbf{h}_{n,1}}) + \prod_n (-q^{-1} \lambda_{n,2} \lambda_{n,1} \mu_{n,1} q^{\mathbf{h}_{n,2}}) .$$

Applying the rule (92) $u^n v^m \mapsto (-q)^{-nm} (-\mathbf{u})^n (-\mathbf{v})^m$, we come to

$$(150) \quad \mathbf{J}(q\mathbf{u}, \mathbf{v}) \mathbf{v}^{-1} = -\mathbf{t}(\mathbf{u}) + \mathbf{v}^{-1} \prod_n (1 - q^{-1} \mathbf{u} \lambda_{n,1} \lambda_{n,2} q^{\mathbf{h}_{n,1} + \mathbf{h}_{n,2}}) + \mathbf{v} \prod_n \mu_{n,1} \mu_{n,2} (q^{\mathbf{h}_{n,1} + \mathbf{h}_{n,2}} - q^{-1} \mathbf{u} \lambda_{n,1} \lambda_{n,2}) .$$

Equation $\langle Q | J(q\mathbf{u}, \mathbf{v}) \mathbf{v}^{-1} | u \rangle = 0$ is exactly Baxter's equation for $\mathcal{U}_q(\widehat{sl}_2)$,

$$(151) \quad Q(u) t(u) = Q(q^{-2} u) \phi(u) + Q(q^2 u) \phi'(u)$$

where

$$(152) \quad \phi(u) = v^{-1} \prod_n (1 - q^{-1} u \lambda_{n,1} \lambda_{n,2} q^{\mathbf{h}_{n,1} + \mathbf{h}_{n,2}}) ,$$

and

$$\phi'(u) = v \prod_n \mu_{n,1} \mu_{n,2} (q^{\mathbf{h}_{n,1} + \mathbf{h}_{n,2}} - q^{-1} u \lambda_{n,1} \lambda_{n,2}) .$$

Condition $Q(0) = 1$ gives (see (149))

$$(153) \quad v^{-1} + v \prod_n \mu_{n,1} \mu_{n,2} q^{\mathbf{h}_{n,1} + \mathbf{h}_{n,2}} = \prod_n \mu_{n,1} q^{\mathbf{h}_{n,1}} + \prod_n \mu_{n,2} q^{\mathbf{h}_{n,2}} ,$$

which has two solutions corresponding to two Baxter's functions Q : the first one is

$$(154) \quad v = v_1 \stackrel{\text{def}}{=} \left(\prod_n \mu_{n,1} q^{\mathbf{h}_{n,1}} \right)^{-1} ,$$

so that $Q = Q_1(u)$ is a polynomial of the degree $\sum_n \mathbf{h}_{n,1} = \mathcal{K}_1$ (see (72)), this value of the degree follows from the second line of eq. (149); the second solution is

$$(155) \quad v = v_2 \stackrel{\text{def}}{=} \left(\prod_n \mu_{n,2} q^{\mathbf{h}_{n,2}} \right)^{-1} ,$$

and corresponding $Q = Q_2(u)$ is the polynomial of the power $\sum_n \mathbf{h}_{n,2} = \mathcal{K}_2$. In some sense, two functions Q correspond to two sheets of classical $q = 1$ spectral hyperelliptic

curve $v^{-1}\phi(u) + v\phi'(u) = t(u)$. Note, we consider now the Bethe Ansatz equations for $\mathcal{U}_q(\widehat{sl}_2)$ chain with arbitrary \mathcal{J}_n , this is the inhomogeneity of highest spin. Six-vertex case corresponds to $\mathcal{J}_n = 1$ for all n .

In the *inhomogeneous* case the spectrum of $t(u)$ follows from (151) and just the condition $Q \neq 0$ (without fixing v and considering the polynomial structure of Q). Let us fix $\mathcal{J}_n = 1$, the six-vertex case and a priori the Fock space representation. Simply substituting $u = (\lambda_{n,1}\lambda_{n,2})^{-1}$ and $u = q^2(\lambda_{n,1}\lambda_{n,2})^{-1}$ into (151) and excluding Q , one comes to

$$(156) \quad t\left(\frac{1}{\lambda_{n,1}\lambda_{n,2}}\right) t\left(\frac{q^2}{\lambda_{n,1}\lambda_{n,2}}\right) = \prod_k q\mu_{k,1}\mu_{k,2} \left(1 - q^2 \frac{\lambda_{k,1}\lambda_{k,2}}{\lambda_{n,1}\lambda_{n,2}}\right) \left(1 - q^{-2} \frac{\lambda_{k,1}\lambda_{k,2}}{\lambda_{n,1}\lambda_{n,2}}\right)$$

with $n = 1, \dots, N$. These N equations and conditions (149) do produce the whole spectrum of $t(u)$ for inhomogeneous six-vertex model.

Equations (156) may be obtained in the other way – via $\overline{Q}(v) = \langle v | \overline{Q} \rangle$. Condition $\overline{Q}(0) = 1$ fixes $u = u_n$,

$$(157) \quad u_n = (\lambda_{n,1}\lambda_{n,2}q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}})^{-1} ,$$

corresponding $\overline{Q} = \overline{Q}_n(v)$ is a polynomial of the power $\mathcal{J}_n = \mathbf{h}_{n,1} + \mathbf{h}_{n,2}$. In the six-vertex case $\mathcal{J}_n = 1$, therefore $\overline{Q}_n(v) = v - \zeta_n$, and equating to zero each v -term of the right of equations (97), one comes to

$$(158) \quad \zeta_n = - \frac{t(q^3 u_n)}{\prod_k q\mu_{k,1}\mu_{k,2}(1 - q^{-2} u_n/u_k)} = - \frac{\prod_k (1 - q^2 u_n/u_k)}{t(qu_n)} .$$

The second equality gives (156).

4.2. $\mathcal{U}_q(\widehat{sl}_M)$ equations. Let the lattice has arbitrary N and M . We will consider (97) for $\langle Q | u \rangle = Q(u)$ and for $\langle u | \overline{Q} \rangle = \overline{Q}(u)$ – the last function was not mentioned before, but it is interesting to discuss what it is.

Equations $\langle Q | J(\mathbf{u}, \mathbf{v}) | u \rangle = 0$ and $\langle u | J(\mathbf{u}, \mathbf{v}) | \overline{Q} \rangle = 0$ read correspondingly

$$(159) \quad \sum_{m=0}^M Q(q^{2m}u)(-v)^m \tau_m(u) = 0 \quad \text{and} \quad \sum_{m=0}^M \tau(q^{-2m}u)(-v)^m \overline{Q}(q^{-2m}u) = 0 ,$$

where

$$(160) \quad \tau_m(u) = \sum_{n=0}^N (-q)^{nm} (-u)^n \mathbf{t}_{nm} ,$$

cf. (5,6). Combinatorially, one can obtain the following summation formulae: analogue of (148)

$$(161) \quad \sum_n u^n t_{n,0} = \prod_n \left(1 + u \prod_m \lambda_{n,m} q^{\mathbf{h}_{n,m}} \right),$$

$$\sum_n u^n t_{n,M} = \prod_n \left(\prod_m \mu_{n,m} q^{\mathbf{h}_{n,m}} + u \prod_m -q^{-1} \lambda_{n,m} \mu_{n,m} \right),$$

and analogue of (149)

$$(162) \quad \sum_m v^m t_{0,m} = \prod_m \left(1 + v \prod_n \mu_{n,m} q^{\mathbf{h}_{n,m}} \right),$$

$$\sum_m v^m t_{N,m} = \prod_m \left(\prod_n \lambda_{n,m} q^{\mathbf{h}_{n,m}} + v \prod_n -q^{-1} \lambda_{n,m} \mu_{n,m} \right).$$

Let us fix notations for the inhomogeneities (cf. (141))

$$(163) \quad u_n = \left(\prod_m \lambda_{nm} q^{\mathbf{h}_{nm}} \right)^{-1}, \quad v_m = \left(\prod_n \mu_{nm} q^{\mathbf{h}_{nm}} \right)^{-1}.$$

It follows from (161),

$$(164) \quad \tau_0(u) = \prod_n \left(1 - \frac{u}{u_n} \right), \quad \tau_M(u) = \prod_m v_m \prod_n \left(1 - q^{-2\mathcal{J}_n} \frac{u}{u_n} \right).$$

For the normalization $Q(0) = \overline{Q}(0) = 1$ both equations (159) are equivalent to the first line of (162):

$$(165) \quad \sum_{m=0}^M (-v)^m \tau_m(0) = \prod_m \left(1 - \frac{v}{v_m} \right) = 0$$

Suppose, $Q(u) \sim u^K$ and $\overline{Q}(u) \sim u^{\overline{K}}$ when $u \rightarrow \infty$. Then (159) and the second line of (162) provides the following conditions for K and \overline{K} :

$$(166) \quad \prod_m \left(1 - q^{2K-2\mathcal{K}_m} \frac{v}{v_m} \right) = 0$$

and

$$(167) \quad \prod_m \left(1 - q^{-2\overline{K}-2N-2\mathcal{K}_m} \frac{v}{v_m} \right) = 0.$$

The charges \mathcal{J}_n and \mathcal{K}_m are given by (72). Equation (165) has the solutions $v = v_m$, $m = 1, 2, 3, \dots, M$. In what follows, we will assume the generic set of μ_{nm} , so that all v_m

are different. Let $v = v_m$ corresponds to $Q(u) = Q_m(u)$ and $\overline{Q}(u) = \overline{Q}_m(u)$. Then (166) and (167) defines the leading $u \rightarrow \infty$ asymptotic

$$(168) \quad Q_m(u) \sim u^{\mathcal{K}_m} \quad \text{and} \quad \overline{Q}_m(u) \sim u^{\overline{\mathcal{K}}_m} ,$$

where

$$(169) \quad \overline{\mathcal{K}}_m = - \sum_n (1 + \mathbf{h}_{n,m}) .$$

Thus, for the Fock space representation all $Q_m(u)$ are polynomials of the power \mathcal{K}_m . Functions $\overline{Q}_m(u)$ are rational functions, below we construct them in the terms of $Q_m(u)$.

Instead of the Fock space representation $\text{Spec}(\mathbf{h}) = 0, 1, 2, \dots, \infty$, one may consider the anti-Fock space, $\text{Spec}(\mathbf{h}) = -1, -2, -3, \dots, -\infty$. Then (134) are to be solved in different way, and as the result $\overline{Q}_m(u)$ become polynomials and $Q_m(u)$ become rational functions.

The dual “T-Q” equations for sl_N correspond to the evident exchange $N \leftrightarrow M$, $u \leftrightarrow v$ and $\mathcal{J} \leftrightarrow \mathcal{K}$.

Turn now to the form of \overline{Q}_m for the Fock space representation. The detailed “arithmetical” consideration of (159) as a set of linear equations allows one to conclude for instance

$$(170) \quad \overline{Q}_M(u) = \frac{W_M(u)}{V(u)} ,$$

where

$$(171) \quad W_M(u) = \det ||Q_j(q^{2i}u)v_j^i||_{i,j=1,\dots,M-1} ,$$

and $\frac{V(u)}{V(q^2u)} = v_1 v_2 \dots v_M \frac{\tau_M(q^M u)}{\tau_0(q^2 u)}$. All the other \overline{Q}_m correspond to (170) and (171) with permuted set of indices of Q_j . Inhomogeneity of v_m is important in this consideration since if $Q_m(0) = 1$, then $W_M(0) = v_1 \dots v_{M-1} \prod_{1 \leq i < j < M} (v_i - v_j)$, and therefore $W_M(u)$ is not zero. As well, for the generic v_m all $Q_m(u)$ are functionally independent, their Wronskian

$$(172) \quad W(u) = \det ||Q_j(q^{2(i-1)}u)v_j^{i-1}||_{i,j=1,\dots,M}$$

obeys

$$(173) \quad \frac{W(u)}{W(q^2u)} = v_1 v_2 \dots v_M \frac{\tau_M(q^M u)}{\tau_0(u)}$$

with the initial condition $W(0) = \prod_{1 \leq i < j \leq M} (v_i - v_j)$, therefore $W(u) \neq 0$.

Using these “arithmetical” considerations, one may express the fundamental transfer matrices $\tau_m(u)$ of sl_M via determinants $\det ||Q_j(q^{2p_i}u)v_j^{p_i}||_{i,j=1,\dots,M}$, where p_i is a subset of $(0, 1, \dots, M)$. Moreover, for a more general sets of p_i , any transfer matrix of sl_M may be expressed as such a determinant [5].

The nested Bethe Ansatz equations (see [6] and e.g. [7]) may be derived from the generalized “T-Q” equations in an “arithmetical” way as well. Let

$$(174) \quad \begin{aligned} \mathcal{Q}_m(u) &= \det ||Q_j(q^{2i}u)v_j^i||_{i,j=1,\dots,m}, \\ \mathcal{P}_m(u) &= \det ||Q_j(q^{2i}u)v_j^i||_{j=1,\dots,m; i=0,2,\dots,m}, \end{aligned}$$

with $\mathcal{Q}_0 = 1$ and $\mathcal{P}_0 = 0$. Function $\mathcal{Q}_m(u)$ is a polynomial of the power $\mathcal{K}_1 + \mathcal{K}_2 + \dots + \mathcal{K}_m$. Equations

$$(175) \quad \mathcal{P}_m(u)\mathcal{Q}_{m-1}(u) = \mathcal{P}_{m-1}(u)\mathcal{Q}_m(u) + \frac{1}{v_m}\mathcal{Q}_{m-1}(q^2u)\mathcal{Q}_m(q^{-2}u)$$

for $m = 1, 2, \dots, M-1$ are just the determinant identities for (174). For $m = M$ the system (175) should be completed by an equation, following from (159):

$$(176) \quad \tau_1(u)\mathcal{Q}_{M-1}(u) = \tau_0(u)\mathcal{P}_{M-1}(u) + v_1 \cdots v_{M-1}\tau_M(u)\mathcal{Q}_{M-1}(q^2u),$$

where τ_0 and τ_M are given by (164). The nested Bethe Ansatz equations are a closed system of algebraic equations for roots of $\mathcal{Q}_m(u)$ following from (175,176).

5. CONCLUSION

This paper has a modest aim just a to give a correct form of “T-Q” equations. We can say nothing about their solution. But we would like to note, that from the point of view of three dimensional models, the thermodynamical limit is the limit $N, M \rightarrow \infty$ with non-singular ratio $N : M$. The nested Bethe Ansatz equations was never investigated in this limit since $\mathcal{Q}_m(u)$ (174) with finite m has a finite number of roots. In addition, an excitation corresponds to a change of the structure of occupation numbers. From sl_M point of view of the previous section, it corresponds not only to a change of \mathcal{K}_m related to the powers of \mathcal{Q} -operators, but as well it corresponds to a change of \mathcal{J}_n which is a change of the sl_M -structure of the nested Bethe Ansatz.

Let us better conclude this paper by a brief comparison of two exactly integrable models in $2+1$ dimensional space-time. From the point of view of the algebra of observables, these models should be called “ q -oscillator model” and “Weyl-algebra model”. The last one is a quantum-mechanical reformulation of Zamolodchikov-Bazhanov-Baxter model of statistical mechanics, which has a long history [8–12]. Both models are based on two

slightly different forms of local linear problem, the linear problem for Weyl-algebra model may be found in any of Refs. [13]. Solution of both classical models may be expressed in terms of algebraic geometry, [3] and [14]. Equations of motion may be understood as a canonical mapping conserving certain symplectic structure, [1] and [15]. Poisson structure allows an immediate quantization, [1] and [13]. Quantum-mechanical integrals of motion may be combined into a direct sum of transfer matrices for fundamental representations of either sl_N or sl_M , [1, 2] and [16]. And finally, the solvability of the models is based on quantized auxiliary linear problem and remarkable features of their operator-valued matrices of coefficients, this paper and [17].

The continuous limit of both classical models may be illustrative. Equations of motion for six fields q_j, q_j^* , $j = 1, 2, 3$, follow from the action

$$(177) \quad A = \int d^3x [q_1^* \partial_1 q_1 + q_2^* \partial_2 q_2 + q_3^* \partial_3 q_3 + V(q^*, q)] ,$$

where for the classical continuous limit of q -oscillator model $V = q_1^* q_2^* q_3^* - q_1 q_2 q_3$, this is nothing but the model of three-wave resonant interaction [18]. For the classical continuous limit of Weyl-algebra model the potential is $V = (q_1^* - q_2)(q_2^* - q_3)(q_3^* - q_1)$ [19].

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